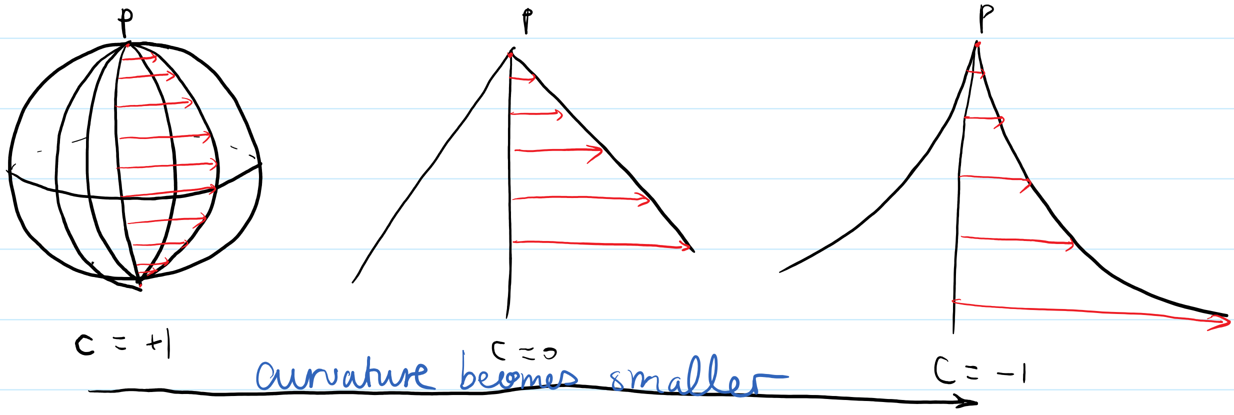
Examples of Jacobi field and cut locus

2020年5月29日 10:33

<u>Rauch comparison Theorem</u>: "smaller curvature leads to larger Jacobi fields"

<u>Examples</u>.

<u>Rough pictures</u>:



A more precise investigation:

Let's look the $\mathbb{S}^n(r)$ with radius r. It has constant sectional curvature $k = 1/r^2$.

Let $\gamma : [0, \pi r]$ be a normal geodesic from $p = \gamma(0)$ to its antipodal point $\gamma(\pi r) = p' = -p$. Denote $T(t) = \dot\gamma(t)$, $t \in [0, \pi r]$

Let $E(t)$ be a parallel vector field along γ such that

$$\langle E(t), T(t)\rangle = 0, \quad \langle E(t), E(t)\rangle = 1, \quad \forall\, t \in [0, \pi r].$$

Let us look for a Jacobi field U of the form $U(t) = f(t)E(t)$ such that $U(0) = 0$ and $U(\pi r) = 0$.

Jacobi equation $\nabla_T \nabla_T U + R(U, T)T = 0$ implies that

$$f''(t)\, E(t) + f(t)\, R(E, T)T = 0$$

and, hence, $f''(t) + f(t)k = 0$, $f(0) = 0$, $f(\pi r) = 0$.

Therefore $f(t) = c \cdot \sin(\sqrt{k}\, t)$ for some const c.

Then we find the following Jacobi field

$$U_k(t) = c \cdot \sin(\sqrt{k}\, t)\, E(t)$$

on $\mathbb{S}^n(r) = \mathbb{S}^n\left(\frac{1}{\sqrt{k}}\right)$

Now, let us compare the norm of Jacobi fields on

$\mathbb{S}^n(r)$, $r = 1, 2, 3, 4, \ldots$

<u>Warning</u>: For a proper comparison, we have to first <u>unify</u>

Warning: For a proper comparison, we have to first <u>unify</u> their initial data: $U(0)$, and, $|\nabla_T U(0)|$.

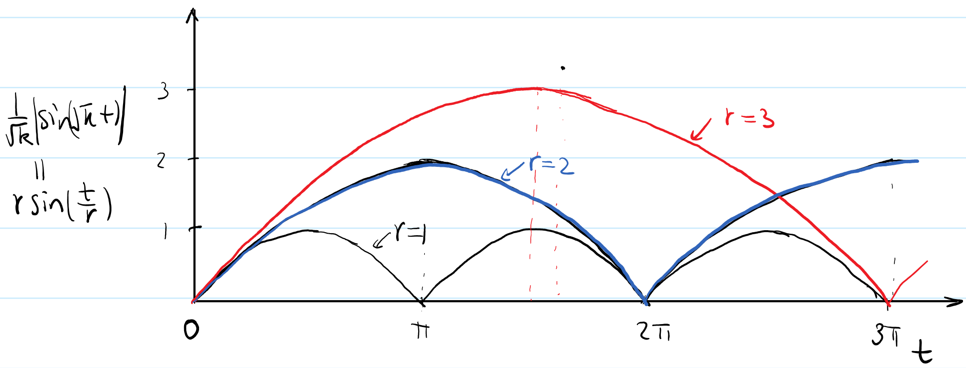
For those $U_k(t) = c \cdot \sin(\sqrt{k}\,t)\, E(t)$, we have

$$U_k(0) = 0, \quad |\nabla_T U_k|(0) = \left| c\sqrt{k}\cos(\sqrt{k}\,t)\, E(t) \big|_{t=0} \right| = |c|\sqrt{k}$$

We can just set $c = \frac{1}{\sqrt{k}}$ s.t. $|\nabla_T U_k|(0) = 1$, and compare the following Jacobi fields (on different spheres)

$$|U_k(t)| = \left| \frac{1}{\sqrt{k}} \sin(\sqrt{k}\,t)\, E(t) \right| = \frac{1}{\sqrt{k}} |\sin\sqrt{k}\,t|.$$

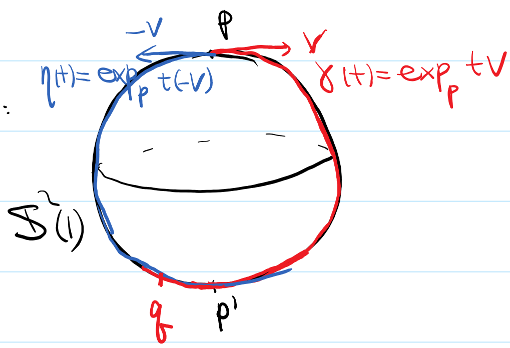
Note that here $U_k(t)$ is Jacobian on $t \in [0, \infty)$!!! since the geodesic γ can be extended to the infinity due to the completeness.



We observe, when r becomes larger, the curvature $k = \frac{1}{r^2}$ becomes smaller, and the norm of Jacobi fields becomes <u>larger</u>, but not always. This is true before the first conjugates points of both geodesics under comparison happen!! Example: the case $r=2$ & $r=3$

<u>Cut locus</u>.

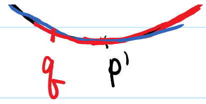
<u>Examples</u>.



Cut locus of p: $C(p) = \{p'\}$

$S^2 \setminus \{p'\}$

$\exp_p : B(0,\pi) \subset T_pM \longrightarrow S^2 \setminus \{p'\}$

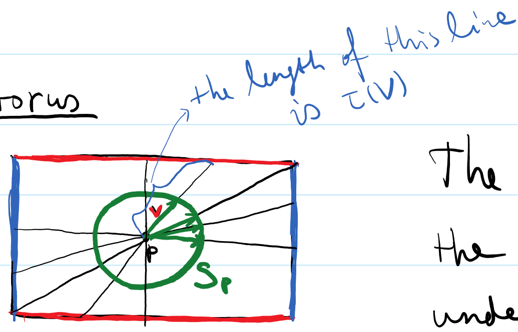


$\exp_p : B(0,\pi) \subset T_pM \to S^2 \setminus \{p'\}$

is a diffeomorphism.

We see, q happens "after" the cut point p' of p along the geodesic γ, meanwhile, q happens "before" the cut point p' of p along the geodesic η.

Flat torus



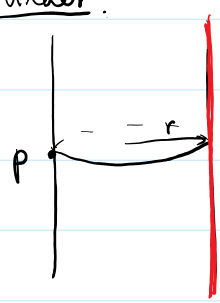
The cut locus $C(p)$ of p is the boundary of the rectangle in the left under "identifications".

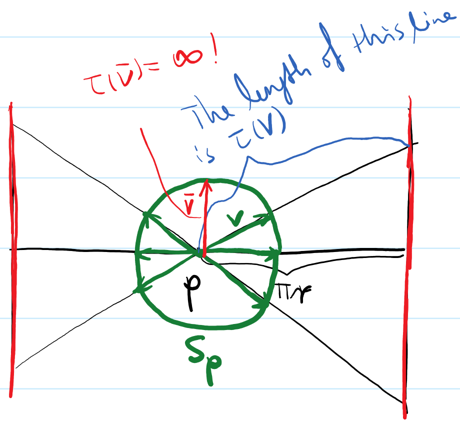
Recall Flat torus has sec $= 0$, and hence has no conjugate points. ⊕ So, for a ~~the~~ cut point q of p, ~~must~~ there must exists at least <u>two</u> minimizing geodesics from p to q.

<u>Observation</u>: If we denote by $S_p := B(0,1) \subset T_pM$.

Then the map
$$\tau : S_p \longrightarrow (0,\infty) \quad \text{is } \underline{\underline{\text{continuous}}},$$
$$V \longmapsto \tau(V)$$
where $\tau(V)$ is a number such that $\exp_p(\tau(V)V)$ is the cut point of p along the normal geodesic $\gamma(t) := \exp_p(tV)$.

Cylinder.





Again, in this case, there are no conjugate points. So for each cut point q of p, there are at least 2 minimizing geodesics.

The cut locus of p is the red line in the left figure.

In this case, let us

The cut locus of p is the red line in the left figure.

In this case, let us investigate the map

$$\tau : S_p \longrightarrow [0, \infty]$$

$\tau(V) = \infty$ if p has no cut point along $\gamma(t) = \exp_p tV$, $t \in (0, \infty)$

We have to include the element ∞ for this case.

Observe: If we assign a topology of $(0, \infty]$ with the basis

$$\{ (a,b),\ (c, \infty],\ \forall a,b,c \in (0,\infty) \}$$

Then, we still have τ is a continuous map !!

In fact, this is a general phenomenon:

Thm: Let M be a complete Rie. manifold, and $p \in M$. Then the map $\tau : S_p \to (0, \infty]$ is continuous.

For the proof, I refer to my lecture notes.

As a corollary, we have: The cut locus $C(p)$ of a point p in a complete Rie. mfld M is a closed subset of M.

Proof of the corollary: Let $p_i \in C(p)$ and p_i converges to $q \in M$ as $i \to \infty$.

Let $\gamma_i = \gamma_i(t)$ be ~~the~~ a minimizing normal geodesic from p to p_i, ~~and~~ we denote $\dot{\gamma}_i(0) = V_i \in T_pM$.

Since $\{V_i\}_{i=1}^{\infty}$ are subset of S_p, which is a compact set, we have, up to a subsequence, V_i converges to $V \in S_p$.

Then we have

$$p_i = \exp_p(\tau(V_i)V_i), \quad \text{and } q = \lim_{i\to\infty} p_i = \lim_{i\to\infty} \exp_p(\tau(V_i)V_i)$$
$$= \exp_p\left(\lim_{i\to\infty} \tau(V_i)V_i\right)$$
$$= \exp_p(\tau(V)V)$$

Hence $q \in C(p)$. $\square$

(M, g) complete

Conclusion: What is a cut locus of a point $p \in M$? It is a closed subset $C(p)$ of M, such that, after "cutting the manifold M along it",

$M \setminus C(p)$ ...

such that, ~~after~~ after "cutting the manifold M along it", $M \setminus C(p)$ becomes an open subset of M which is diffeomorphic to an Euclidean domain via $(\exp_p)^{-1}$.